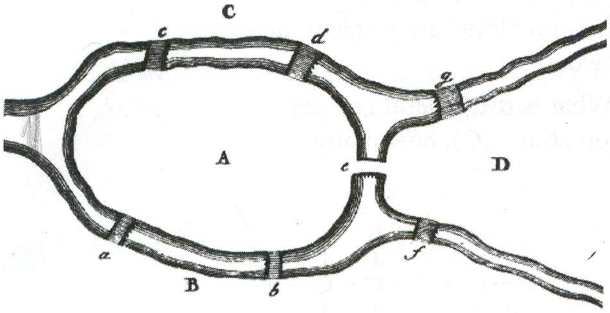
Exploration

The Seven Bridges of Königsberg

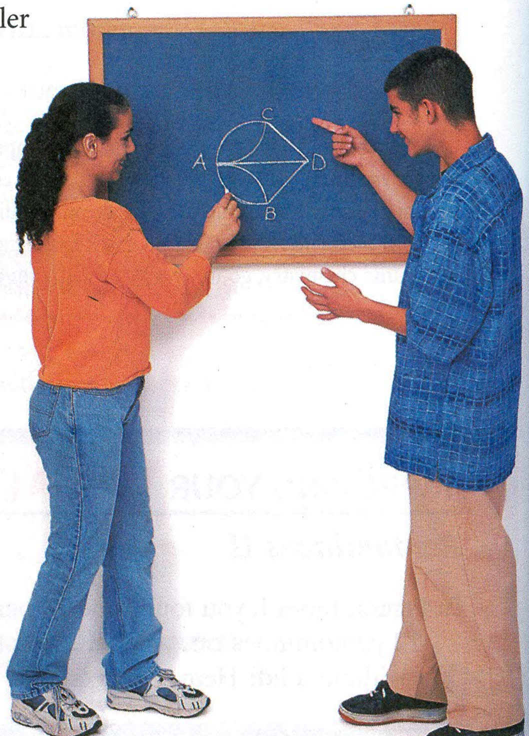
Leonhard Euler

The River Pregel runs through the university town of Königsberg (now Kaliningrad in Russia). In the middle of the river are two islands connected to each other and to the rest of the city by seven bridges. Many years ago, a tradition developed among the townspeople of Königsberg. They challenged one another to make a round trip over all seven bridges, walking over each bridge once and only once before returning to the starting point.



The seven bridges of Königsberg

For a long time no one was able to do it, and yet no one was able to show that it couldn't be done. In 1735, they finally wrote to Leonhard Euler (1707–1783), a Swiss mathematician, asking for his help on the problem. Euler (pronounced "oyler") reduced the problem to a network of paths connecting the two sides of the rivers C and B, and the two islands A and D, as shown in the network at right. Then Euler demonstrated that the task is impossible.



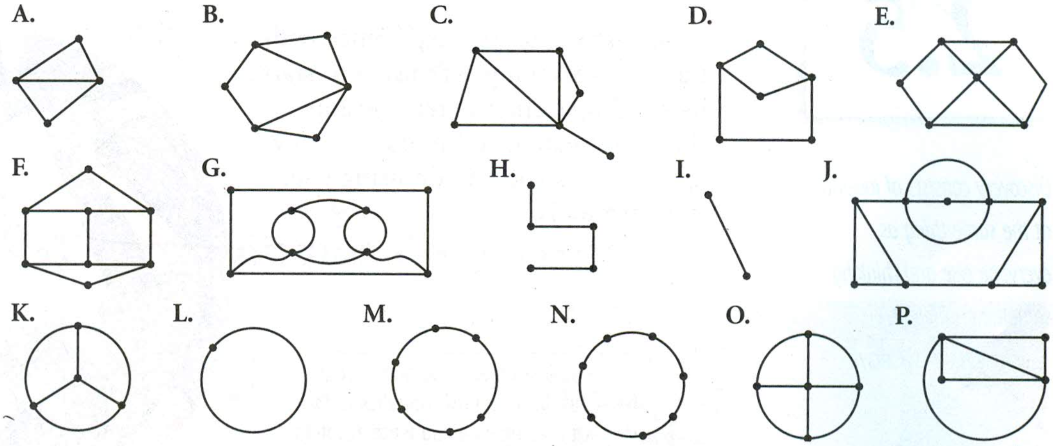
In this activity you will work with a variety of networks to see if you can come up with a rule to find out whether a network can or cannot be "traveled."

Activity

Traveling Networks

A collection of points connected by paths is called a **network.** When we say a network can be traveled, we mean that the network can be drawn with a pencil without lifting the pencil off the paper and without retracing any paths. (Points can be passed over more than once.)

Step 1 | Try these networks and see which ones can be traveled and which are impossible to travel.



Which networks were impossible to travel? Are they impossible or just difficult? How can you be sure? As you do the next few steps, see if you can find the reason why some networks are impossible to travel.

Step 2 | Draw the River Pregel and the two islands shown on the first page of this exploration. Draw an eighth bridge so that you can travel over all the bridges exactly once if you start at point *C* and end at point *B*.

Step 3 | Draw the River Pregel and the two islands. Can you draw an eighth bridge so that you can travel over all the bridges exactly once, starting and finishing at the same point? How many solutions can you find?

Step 4 | Euler realized that it is the points of intersection that determine whether a network can be traveled. Each point of intersection is either "odd" or "even."

Odd points | Even points

Did you find any networks that have only one odd point? Can you draw one? Try it. How about three odd points? Or five odd points? Can you create a network that has an odd number of odd points? Explain why or why not.

Step 5 | How does the number of even points and odd points affect whether a network can be traveled?

Conjecture

A network can be traveled if ___?___.

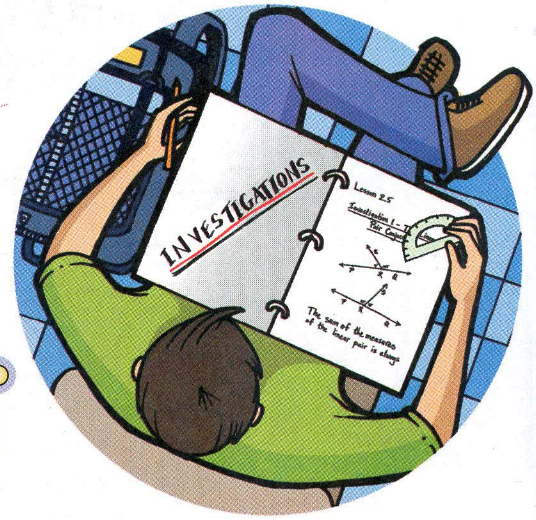
LESSON 2.5

Angle Relationships

Discovery consists of looking at the same thing as everyone else and thinking something different.

ALBERT SZENT-GYÖRGYI

Now that you've had experience with inductive reasoning, let's use it to start discovering geometric relationships. This investigation is the first of many investigations you will do using your geometry tools.



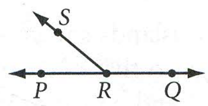
Create an investigation section in your notebook. Include a title and illustration for each investigation and write a statement summarizing the results of each one.

Investigation 1

The Linear Pair Conjecture

You will need

- a protractor



Step 1 On a sheet of paper, draw $\overleftrightarrow{PQ}$ and place a point R between P and Q. Choose another point S not on $\overleftrightarrow{PQ}$ and draw $\overrightarrow{RS}$. You have just created a linear pair of angles. Place the "zero edge" of your protractor along $\overleftrightarrow{PQ}$. What do you notice about the sum of the measures of the linear pair of angles?

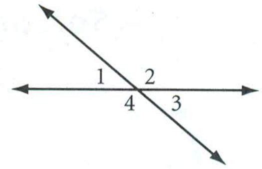
Step 2 Compare your results with those of your group. Does everyone make the same observation? Complete the statement.

Linear Pair Conjecture C-1

If two angles form a linear pair, then $\underline{\ ?\ }$.

The important conjectures have been given a name and a number. Start a list of them in your notebook. The Linear Pair Conjecture (C-1) and the Vertical Angles Conjecture (C-2) should be the first entries on your list. Make a sketch for each conjecture.

In the previous investigation you discovered the relationship between a linear pair of angles, such as $\angle 1$ and $\angle 2$ in the diagram at right.

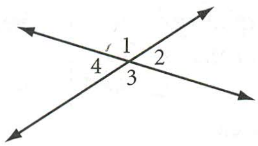


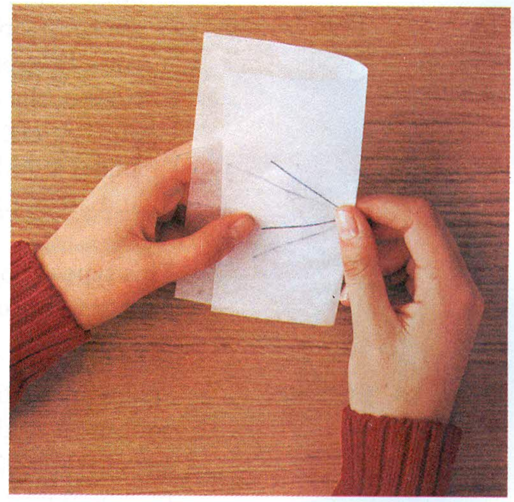
You will discover the relationship between vertical angles, such as $\angle 1$ and $\angle 3$, in the next investigation.

Investigation 2
Vertical Angles Conjecture

You will need

- a protractor
- patty paper





Step 1 Draw two intersecting lines onto patty paper or tracing paper. Label the angles as shown. Which angles are vertical angles?

Step 2 Fold the paper so that the vertical angles lie over each other. What do you notice about their measures?

Step 3 Repeat this investigation with another pair of intersecting lines.

Step 4 Compare your results with the results of others. Complete the statement.

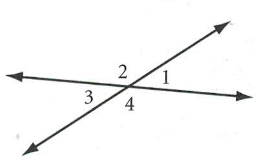
Vertical Angles Conjecture C-2

If two angles are vertical angles, then $\underline{?}$.

You used inductive reasoning to discover both the Linear Pair Conjecture and the Vertical Angles Conjecture. Are they related in any way? That is, if we accept the Linear Pair Conjecture as true, can we use deductive reasoning to show that the Vertical Angles Conjecture must be true?

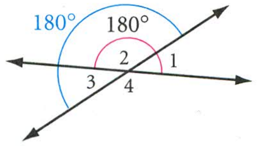
EXAMPLE

The Linear Pair Conjecture states that every linear pair adds up to 180°. Using this conjecture and the diagram, write a logical argument explaining why $\angle 1$ must be congruent to $\angle 3$.



▸ *Solution*

You can see that the measures of $\angle 1$ and $\angle 2$ add up to 180°, and that the measures of $\angle 3$ and $\angle 2$ also add up to 180°. Using algebra, we can write a logical argument to show that $\angle 1$ and $\angle 3$ must be congruent.



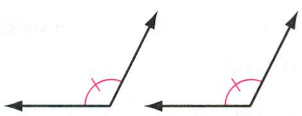
According to the Linear Pair Conjecture, $m\angle 1 + m\angle 2 = 180°$ and $m\angle 2 + m\angle 3 = 180°$. By substituting $m\angle 2 + m\angle 3$ for 180° in the first statement, you get $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$. By the subtraction property of equality, you can subtract $m\angle 2$ from both sides of the equation to get $m\angle 1 = m\angle 3$. Therefore, vertical angles 1 and 3 have equal measures and are congruent.

Here are the algebraic steps:

$m\angle 2 + m\angle 3 = 180°$
$m\angle 1 + m\angle 2 = 180°$
$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$
thus $m\angle 1 = m\angle 3$
therefore $\angle 1 \cong \angle 3$

This type of logical explanation, written as a paragraph, is called a **paragraph proof.**

Now consider another idea. You discovered the Vertical Angles Conjecture: If two angles are vertical angles, then they are congruent. Does that also mean that all congruent angles are vertical angles? The **converse** of an "if-then" statement switches the "if" and "then" parts. The converse of the Vertical Angles Conjecture may be stated: If two angles are congruent, then they are vertical angles. Is this converse statement true? Remember that if you can find even one counterexample, like the diagram below, then the statement is false.



Therefore, the converse of the Vertical Angles Conjecture is false.

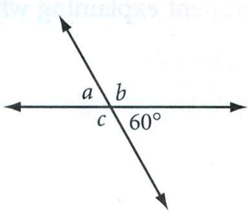
EXERCISES

You will need

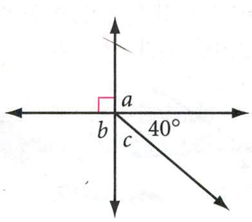
Geometry software for Exercise **12**

Without using a protractor, but with the aid of your two new conjectures, find the measure of each lettered angle in Exercises 1–5. Copy the diagrams so that you can write on them. List your answers in alphabetical order.

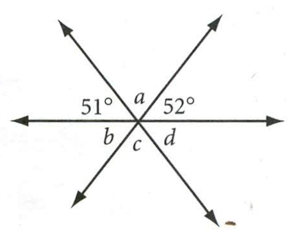
1.



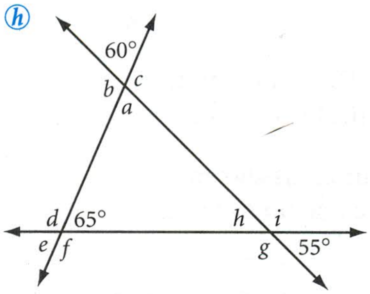
2.



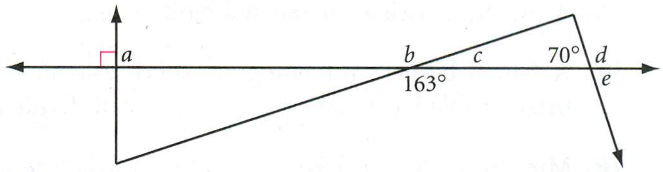
3.



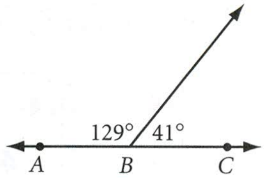
4. ⓗ



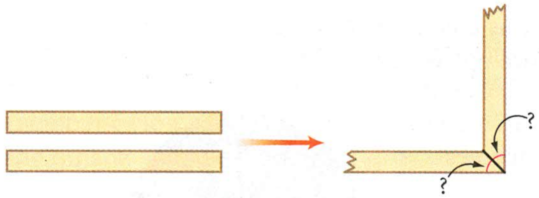
5.



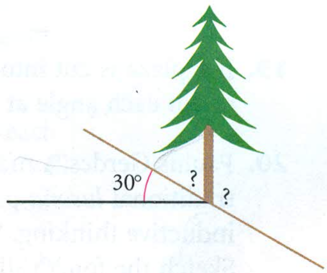
6. Points A, B, and C at right are collinear. What's wrong with this picture?



7. Yoshi is building a cold frame for his plants. He wants to cut two wood strips so that they'll fit together to make a right-angled corner. At what angle should he cut ends of the strips?



8. A tree on a 30° slope grows straight up. What are the measures of the greatest and smallest angles the tree makes with the hill? Explain.



9. You discovered that if a pair of angles is a linear pair then the angles are supplementary. Does that mean that all supplementary angles form a linear pair of angles? Is the converse true? If not, sketch a counterexample.

10. If two congruent angles are supplementary, what must be true of the two angles? Make a sketch, then complete the following conjecture: If two angles are both congruent and supplementary, then __?__.

11. Using algebra, write a paragraph proof that explains why the conjecture from Exercise 10 is true.

12. ***Technology*** Use geometry software to construct two intersecting lines. Measure a pair of vertical angles. Use **calculate** to find the ratio of their measures. What is the ratio? Drag one of the lines. Does the ratio ever change? Does this demonstration convince you that the Vertical Angles Conjecture is true? Does it explain why it is true?

Review

For Exercises 13–17, sketch, label, and mark the figure.

13. Scalene obtuse triangle PAT with $PA = 3$ cm, $AT = 5$ cm, and $\angle A$ an obtuse angle

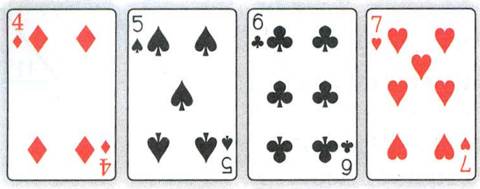
14. A quadrilateral that has rotational symmetry but not reflectional symmetry

15. A circle with center at O and radii $\overline{OA}$ and $\overline{OT}$ creating a minor arc $\overset{\frown}{AT}$

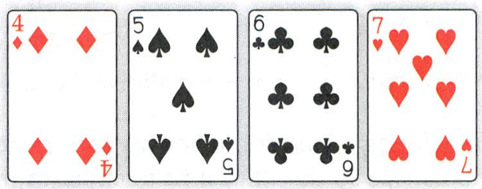
16. A pyramid with an octagonal base

17. A 3-by-4-by-6-inch rectangular solid rests on its smallest face. Draw lines on the three visible faces, showing how you can divide it into 72 identical smaller cubes.

18. Miriam the Magnificent placed four cards face up (the first four cards shown below). Blindfolded, she asked someone from her audience to come up to the stage and turn one card 180°.



Before turn

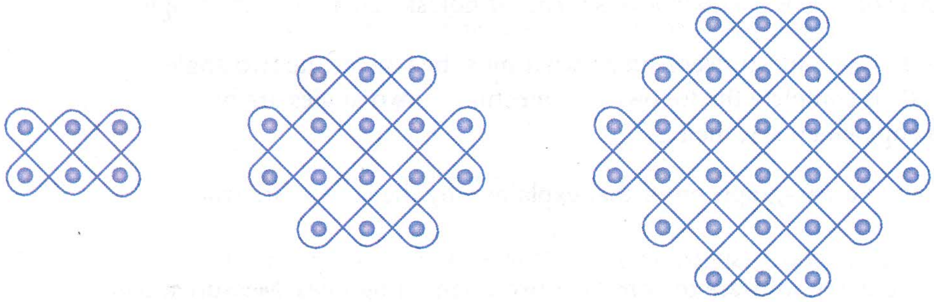


After turn

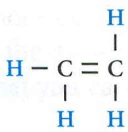
Miriam removed her blindfold and claimed she was able to determine which card was turned 180°. What is her trick? Can you figure out which card was turned? Explain.

19. If a pizza is cut into 16 equal pieces, how many degrees are in each angle at the center of the pizza?

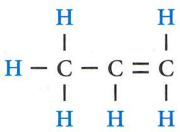
20. Paulus Gerdes, a mathematician from Mozambique, uses traditional *lusona* patterns from Angola to practice inductive thinking. Shown below are three *sona* designs. Sketch the fourth *sona* design, assuming the pattern continues.



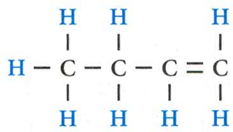
21. Hydrocarbon molecules in which all the bonds between the carbon atoms are single bonds except one double bond are called *alkenes*. The first three alkenes are modeled below.



Ethene
(C_2H_4)



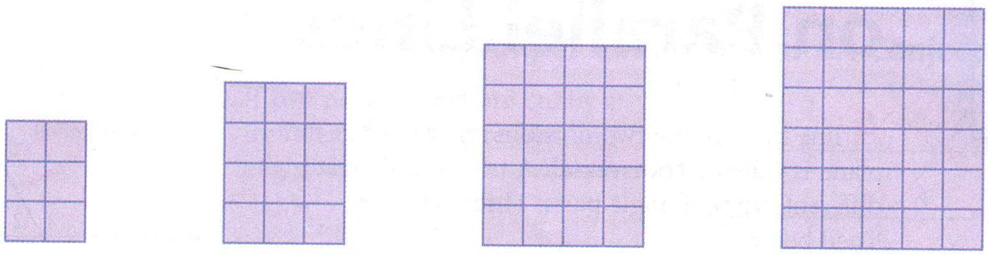
Propene
(C_3H_6)



Butene
(C_4H_8)

Sketch the alkene with eight carbons in the chain. What is the general rule for alkenes $(C_nH_?)$? In other words, if there are n carbon atoms (C), how many hydrogen atoms (H) are in the alkene?

22. If the pattern of rectangles continues, what is the rule for the perimeter of the nth rectangle, and what is the perimeter of the 200th rectangle?



Perimeter in a rectangular pattern

Rectangle	1	2	3	4	5	6	...	n	...	200
Perimeter of rectangle	10	14	18				...		...	

23. The twelfth grade class of 80 students is assembled in a large circle on the football field at halftime. Each student is connected by a string to each of the other class members. How many pieces of string are necessary to connect each student to all the others? (h)

24. If you draw 80 lines on a piece of paper so that no 2 lines are parallel to each other and no 3 lines pass through the same point, how many intersections will there be? (h)

25. If there are 20 couples at a party, how many different handshakes can there be between pairs of people? Assume that the two people in each couple do not shake hands with each other. (h)

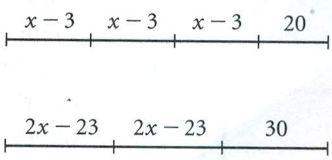
26. If a polygon has 24 sides, how many diagonals are there from each vertex? How many diagonals are there in all?

27. If a polygon has a total of 560 diagonals, how many vertices does it have? (h)

IMPROVING YOUR ALGEBRA SKILLS

Number Line Diagrams

1. The two segments at right have the same length. Translate the number line diagram into an equation, then solve for the variable x.



2. Translate this equation into a number line diagram.
$2(x + 3) + 14 = 3(x - 4) + 11$