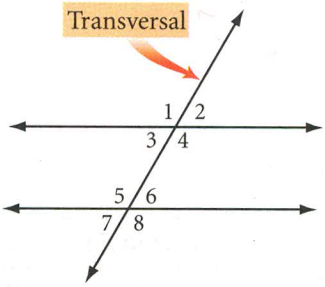
LESSON

2.6 Special Angles on Parallel Lines

The greatest mistake you can make in life is to be continually fearing that you will make one.

ELLEN HUBBARD

A line intersecting two or more other lines in the plane is called a **transversal.** A transversal creates different types of angle pairs. Three types are listed below.

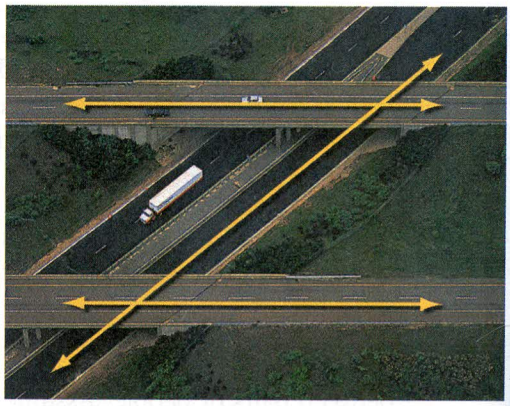


One pair of **corresponding angles** is ∠1 and ∠5. Can you find three more pairs of corresponding angles?

One pair of **alternate interior angles** is ∠3 and ∠6. Do you see another pair of alternate interior angles?

One pair of **alternate exterior angles** is ∠2 and ∠7. Do you see the other pair of alternate exterior angles?

When parallel lines are cut by a transversal, there is a special relationship among the angles. Let's investigate.

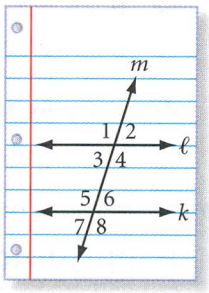


Investigation 1
Which Angles Are Congruent?

You will need

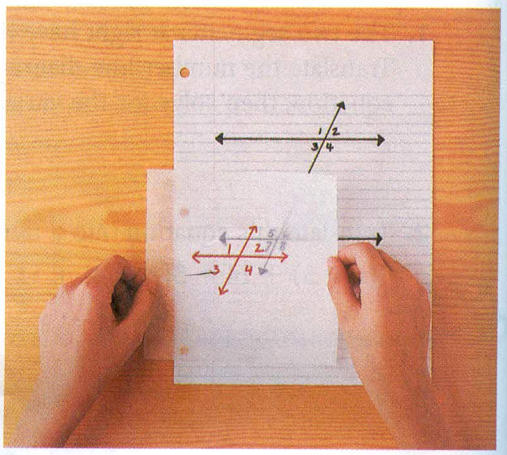
- lined paper or a straightedge
- patty paper
- a protractor

Using the lines on your paper as a guide, draw a pair of parallel lines. Or use both edges of your ruler or straightedge to create parallel lines. Label them k and ℓ. Now draw a transversal that intersects the parallel lines. Label the transversal m, and label the angles with numbers, as shown at right.



Step 1 Place a piece of patty paper over the set of angles 1, 2, 3, and 4. Copy the two intersecting lines m and ℓ and the four angles onto the patty paper.

Slide the patty paper down to the intersection of lines m and k, and compare angles 1 through 4 with each of the corresponding angles 5 through 8. What is the relationship between corresponding angles? Alternate interior angles? Alternate exterior angles?

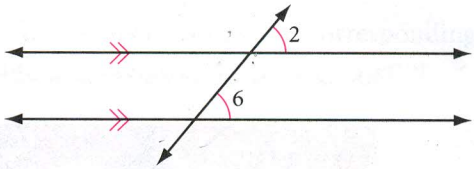


Compare your results with the results of others in your group and complete the three conjectures below.

Corresponding Angles Conjecture, or CA Conjecture

C-3a

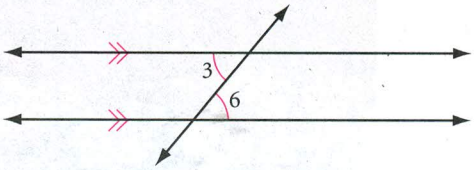
If two parallel lines are cut by a transversal, then corresponding angles are ?.



Alternate Interior Angles Conjecture, or AIA Conjecture

C-3b

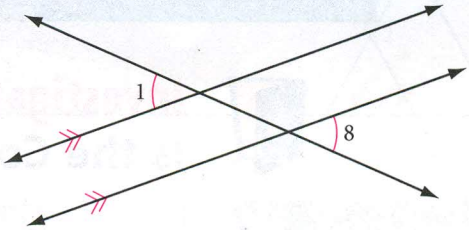
If two parallel lines are cut by a transversal, then alternate interior angles are ?.

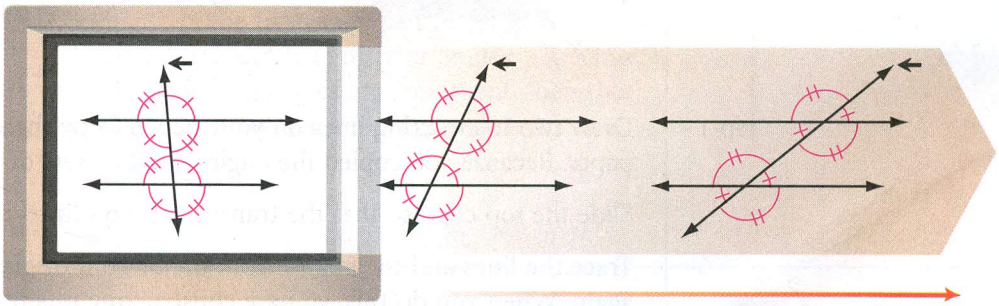


Alternate Exterior Angles Conjecture, or AEA Conjecture

C-3c

If two parallel lines are cut by a transversal, then alternate exterior angles are ?.





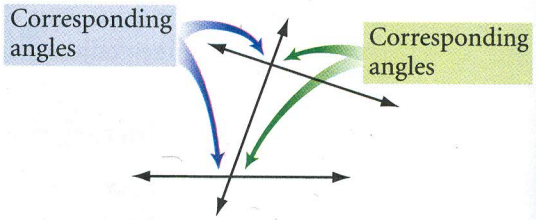
The three conjectures you wrote can all be combined to create a Parallel Lines Conjecture, which is really three conjectures in one.

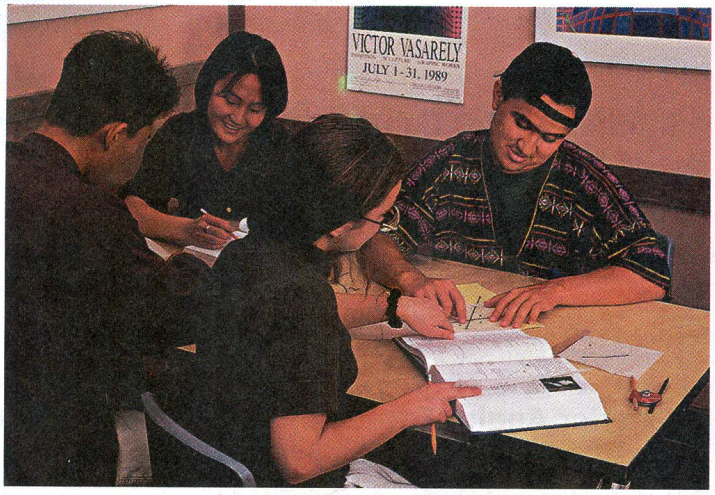
Parallel Lines Conjecture

C-3

If two parallel lines are cut by a transversal, then corresponding angles are ?, alternate interior angles are ?, and alternate exterior angles are ?.

Step 2 What happens if the lines you start with are not parallel? Check whether your conjectures will work with nonparallel lines.



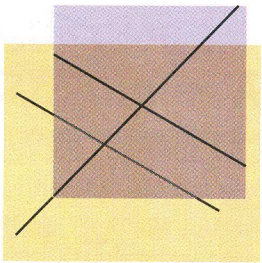


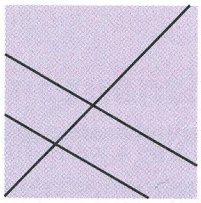
What about the converse of each of your conjectures? Suppose you know that a pair of corresponding angles, or alternate interior angles, is congruent. Will the lines be parallel? Is it possible for the angles to be congruent but for the lines not to be parallel?

Investigation 2
Is the Converse True?

You will need

- lined paper or a straightedge
- patty paper
- a protractor



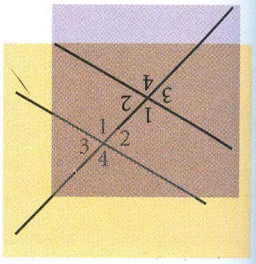


Step 1 Draw two intersecting lines on your paper. Copy these lines onto a piece of patty paper. Because you copied the angles, the two sets of angles are congruent.

Slide the top copy so that the transversal stays lined up.

Trace the lines and the angles from the bottom original onto the patty paper again. When you do this, you are constructing sets of congruent corresponding angles. Mark the congruent angles.

Are the two lines parallel? You can test to see if the distance between the two lines remains the same, which guarantees that they will never meet.



Step 2 Repeat Step 1, but this time rotate your patty paper 180° so that the transversal lines up again. What kinds of congruent angles have you created? Trace the lines and angles and mark the congruent angles. Are the lines parallel? Check them.

Step 3 Compare your results with those of your group. If your results do not agree, discuss them until you have convinced each other. Complete the conjecture below and add it to your conjecture list.

Converse of the Parallel Lines Conjecture

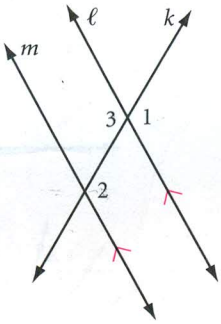
C-4

If two lines are cut by a transversal to form pairs of congruent corresponding angles, congruent alternate interior angles, or congruent alternate exterior angles, then the lines are ?.

You used inductive reasoning to discover all three parts of the Parallel Lines Conjecture. However, if you accept any one of them as true, you can use deductive reasoning to show that the others are true.

EXAMPLE

Suppose we assume that the Vertical Angles Conjecture is true. Write a paragraph proof showing that if corresponding angles are congruent, then the Alternate Interior Angles Conjecture is true.



▶ Solution

Paragraph Proof

Lines ℓ and m are parallel and intersected by transversal k. Pick any two alternate interior angles, such as $\angle 2$ and $\angle 3$. According to the Corresponding Angles Conjecture, $\angle 2 \cong \angle 1$. And, according to the Vertical Angles Conjecture, $\angle 1 \cong \angle 3$. Substitute $\angle 3$ for $\angle 1$ in the first statement to get $\angle 2 \cong \angle 3$. But $\angle 2$ and $\angle 3$ are alternate interior angles. Therefore, if the corresponding angles are congruent, then the alternate interior angles are congruent. ■

It helps to visualize each statement and to mark all congruences you know on your paper.

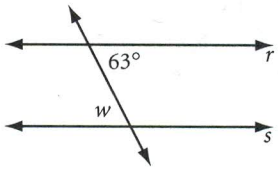
Here are the algebraic steps:

$\angle 2 \cong \angle 1$
$\angle 3 \cong \angle 1$
So $\angle 2 \cong \angle 3$

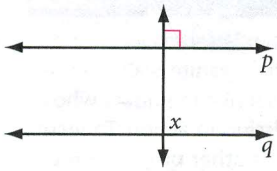
EXERCISES

Use your new conjectures in Exercises 1–6. A small letter in an angle represents the angle measure.

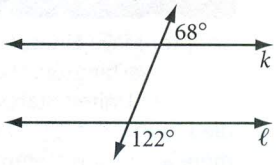
1. $r \parallel s$
$w =$?



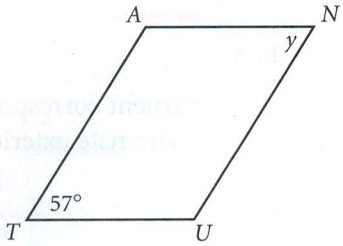
2. $p \parallel q$
$x =$?



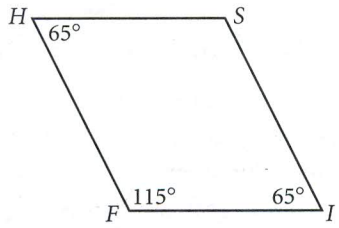
3. Is line k parallel to line ℓ?



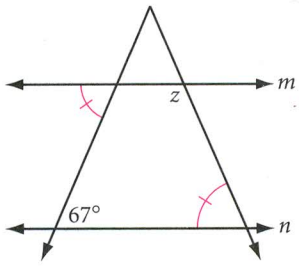
4. Quadrilateral *TUNA* is a parallelogram.
$y = \underline{?}$ ⓗ



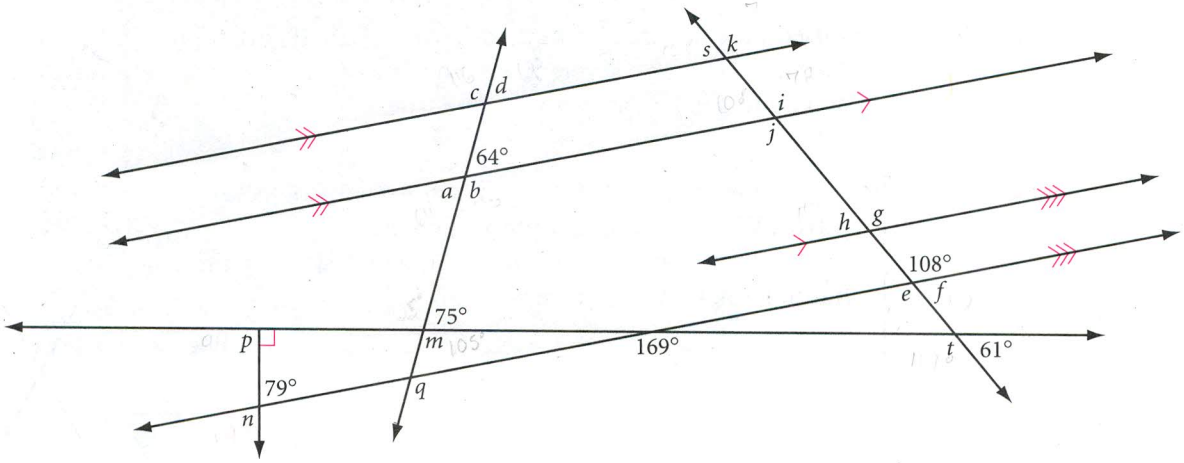
5. Is quadrilateral *FISH* a parallelogram?



6. $m \parallel n$
$z = \underline{?}$ ⓗ

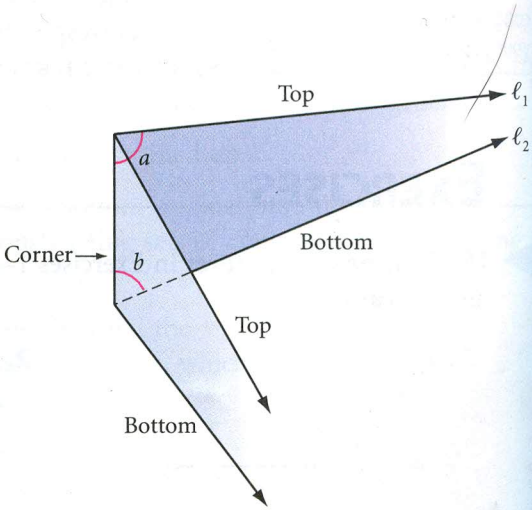


7. Trace the diagram below. Calculate each lettered angle measure. ⓗ

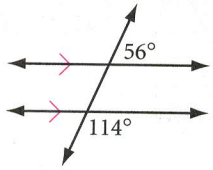


8. You've seen before how parallel lines appear to meet in the distance. Let's look at the converse of this effect: The top and the bottom of the Vietnam Veterans Memorial Wall appear to be parallel because they appear to meet so far in the distance. Consider the diagram of the corner of the memorial, shown below. You know that line ℓ_1 and line ℓ_2 eventually meet. Is the blue shaded portion of the wall a rectangle? Write a paragraph proof explaining why it is or is not a rectangle.

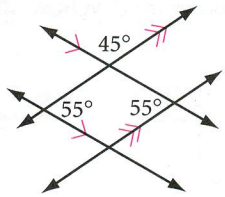
Sculptor Maya Lin designed the Vietnam Veterans Memorial Wall in Washington, D.C. Engraved in the granite wall are the names of United States armed forces service members who died in the Vietnam War or remain missing in action. To learn more about the Memorial Wall and Lin's other projects, visit **www.keymath.com/DG**



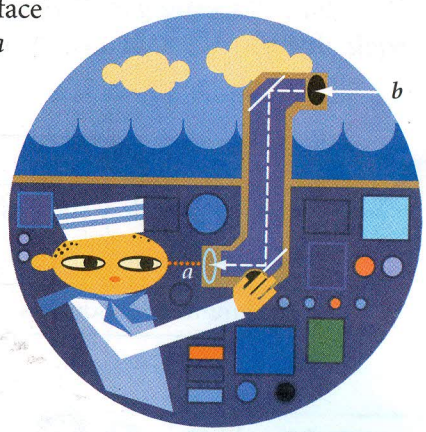
9. What's wrong with this picture?



10. What's wrong with this picture?



11. A periscope permits a sailor on a submarine to see above the surface of the ocean. This periscope is designed so that the line of sight a is parallel to the light ray b. The middle tube is perpendicular to the top and bottom tubes. What are the measures of the incoming and outgoing angles formed by the light rays and the mirrors in this periscope? Are the surfaces of the mirrors parallel? How do you know?



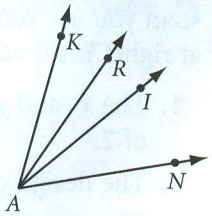
Review

12. What type (or types) of triangle has one or more lines of symmetry?

13. What type (or types) of quadrilateral has only rotational symmetry? (h)

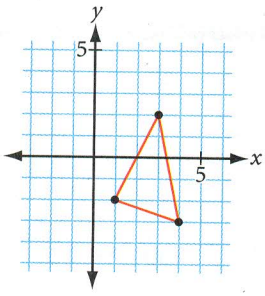
14. If D is the midpoint of $\overline{AC}$ and C is the midpoint of $\overline{BD}$, what is the length of $\overline{AB}$ if $BD = 12$ cm?

15. If $\overrightarrow{AI}$ is the angle bisector of $\angle KAN$ and $\overrightarrow{AR}$ is the angle bisector of $\angle KAI$, what is $m\angle RAN$ if $m\angle RAK = 13°$?

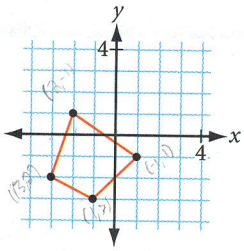


For Exercises 16–18, draw each polygon on graph paper. Relocate the vertices according to the rule. Connect the new points to form a new polygon. Describe what happened to the figure. Is the new polygon congruent to the original?

16. Rule: Subtract 1 from each x-coordinate. (h)



17. Rule: Reverse the sign of each x- and y-coordinate.

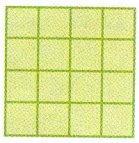


18. Rule: Switch the x- and y-coordinates. Pentagon *LEMON* with vertices:

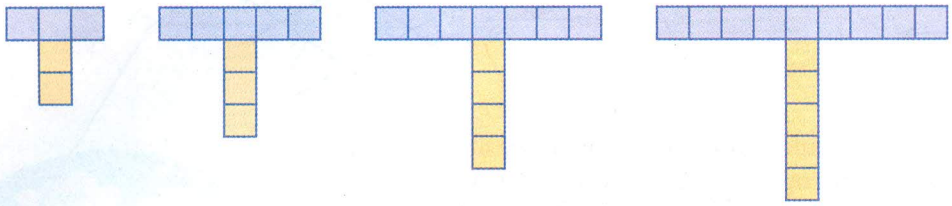
$L(-4, 2)$
$E(-4, -3)$
$M(0, -5)$
$O(3, 1)$
$N(-1, 4)$

19. If everyone in the town of Skunk's Crossing (population 84) has a telephone, how many different lines are needed to connect all the phones to each other?

20. How many squares of all sizes are in a 4-by-4 grid of squares? (There are more than 16!) (h)



21. Assume the pattern of blue and yellow shaded **T**'s continues. Copy and complete the table for blue shaded and yellow shaded squares and for the total number of squares. (h)

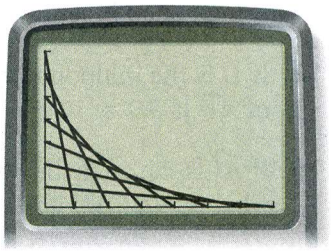


The T-formation

Figure number	1	2	3	4	5	6	...	n	...	35
Number of yellow squares	2						...		...	
Number of blue squares	3						...		...	
Total number of squares	5						...		...	

project

LINE DESIGNS



Can you use your graphing calculator to make the line design shown at right? You'll need to recall some algebra. Here are some hints.

1. The x- and y-ranges are set to minimums of 0 and maximums of 7.
2. The design consists of the graphs of seven lines.
3. The equation for one of the lines is $y = -\frac{1}{7}x + 1$.
4. There's a simple pattern in the slopes and y-intercepts of the lines.

You're on your own from here. Experiment! Then create a line design of your own and write the equations for it.

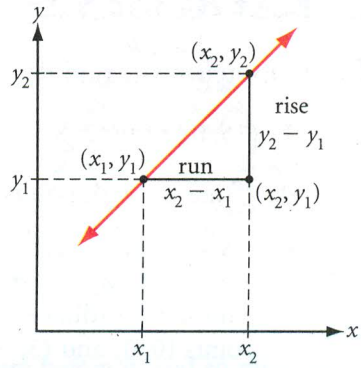
Slope

The slope of a line is a measure of its steepness. Measuring slope tells us the steepness of a hill, the pitch of a roof, or the incline of a ramp. On a graph, slope can tell us the rate of change, or speed.

To calculate slope, you find the ratio of the vertical distance to the horizontal distance traveled, sometimes referred to as "rise over run."

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}}$$

One way to find slope is to use a **slope triangle.** Then use the coordinates of its vertices in the formula.



Slope Formula

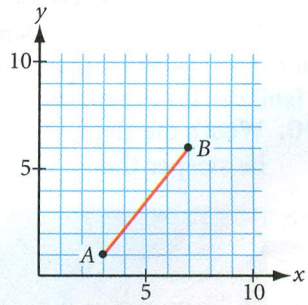
The slope m of a line (or segment) through two points with coordinates (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_2 - x_1 \neq 0$.

EXAMPLE

Draw the slope triangle and find the slope for $\overline{AB}$.



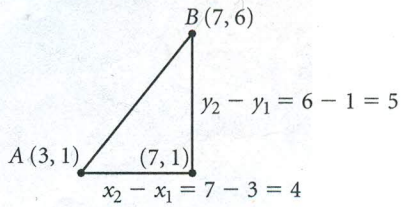
▶ Solution

Draw the horizontal and vertical sides of the slope triangle below the line. Use them to calculate the side lengths.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 1}{7 - 3} = \frac{5}{4}$$

Note that if the slope triangle is *above* the line, you subtract the numbers in reverse order, but still get the same result.

$$m = \frac{1 - 6}{3 - 7} = \frac{-5}{-4} = \frac{5}{4}$$



The slope is positive when the line goes up from left to right. The slope is negative when the line goes down from left to right. When is the slope 0? What is the slope of a vertical line?

EXERCISES

In Exercises 1–3 find the slope of the line through the given points.

1. (16, 0) and (12, 8)

2. (−3, −4) and (−16, 8)

3. (5.3, 8.2) and (0.7, −1.5)

4. A line through points (−5, 2) and (2, y) has a slope of 3. Find y.

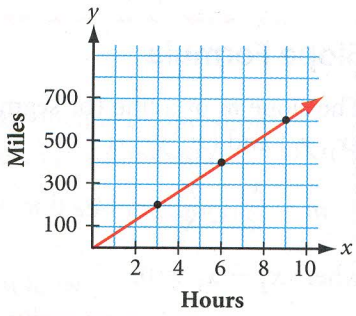
5. A line through points (x, 2) and (7, 9) has a slope of $\frac{7}{3}$. Find x.

6. Find the coordinates of three more points that lie on the line passing through the points (0, 0) and (3, −4). Explain your method.

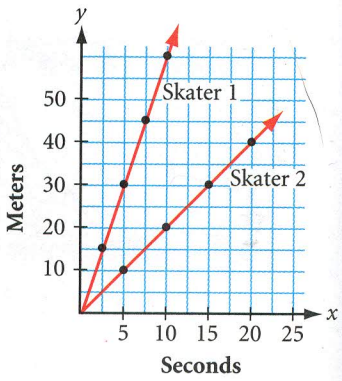
7. What is the speed, in miles per hour, represented by Graph A?

8. From Graph B, which in-line skater is faster? How much faster?

9. The grade of a road is its slope, given as a percent. For example, a road with a 6% grade has slope $\frac{6}{100}$. It rises 6 feet for every 100 feet of horizontal run. Describe a 100% grade. Do you think you could drive up it? Could you walk up it? Is it possible for a grade to be greater than 100%?

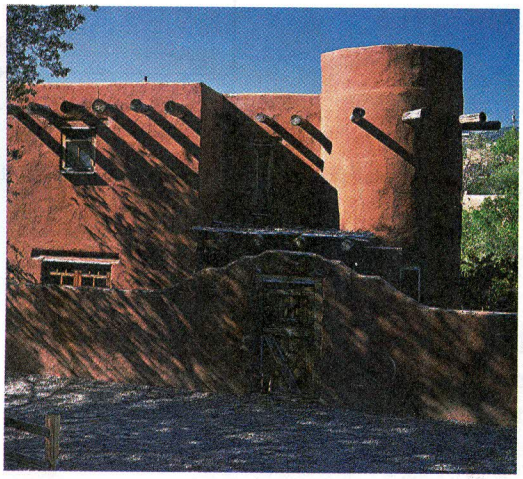


Graph A

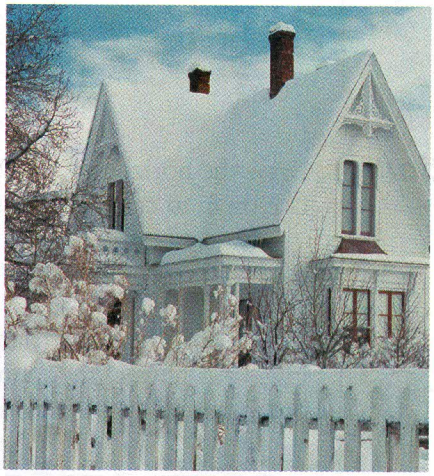


Graph B

10. What's the slope of the roof on the adobe house? Why might a roof in Connecticut be steeper than a roof in the desert?



Adobe house, New Mexico



Pitched-roof house, Connecticut

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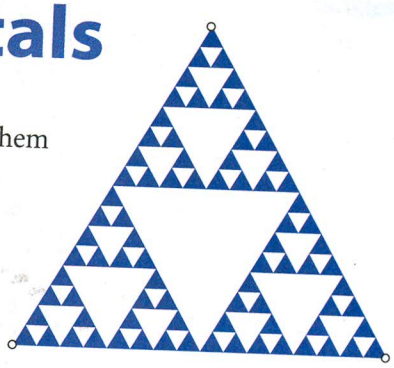
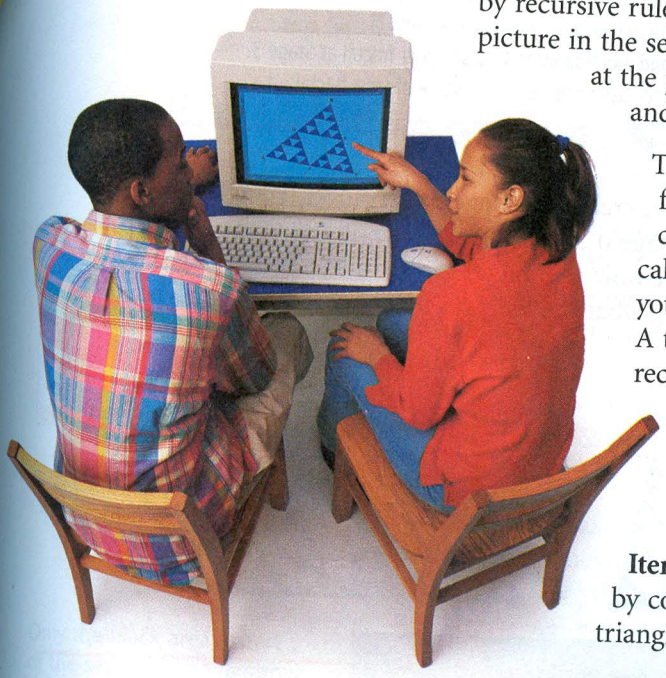
Exploration

Patterns in Fractals

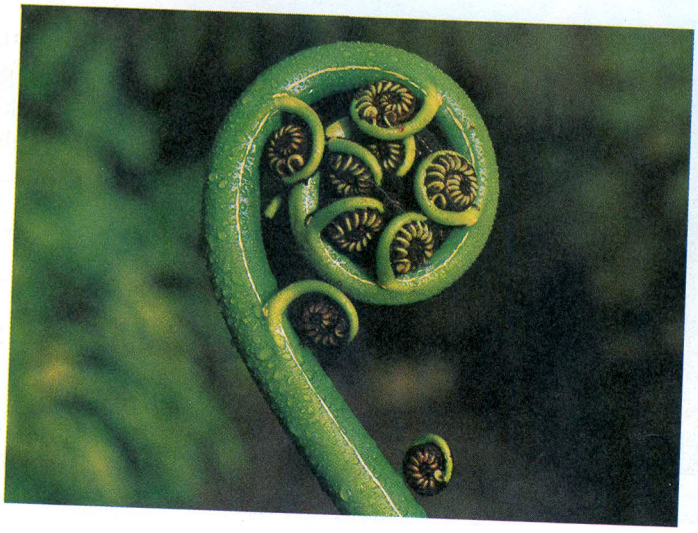


In Lesson 2.1, you discovered patterns and used them to continue number sequences. In most cases, you found each term by applying a rule to the term before it. Such rules are called **recursive rules.** Some picture patterns are also generated by recursive rules. You find the next picture in the sequence by looking at the picture before it and comparing that to the picture before it, and so on.



The Geometer's Sketchpad® can repeat a recursive rule on a figure using a command called **Iterate.** Using **Iterate,** you can create the initial stages of fascinating geometric figures called **fractals.** Fractals have **self-similarity,** meaning that if you zoom in on a part of the figure, it looks like the whole. A true fractal would need infinitely many applications of the recursive rule. In this exploration, you'll use **Iterate** to create the first few stages of a fractal called the **Sierpiński triangle.**

In this procedure you will construct a triangle, its interior, and midpoints on its sides. Then you will use **Iterate** to repeat the process on three outer triangles formed by connecting the midpoints and vertices of the original triangle.



This fern frond illustrates self-similarity. Notice how each curled leaf resembles the shape of the entire curled frond.

Activity
The Sierpiński Triangle

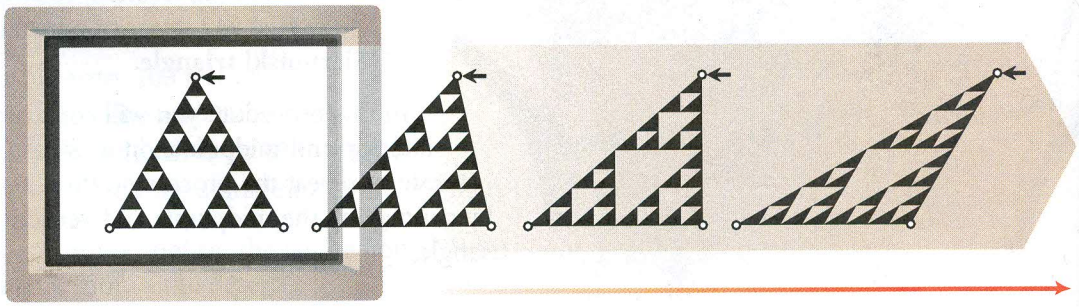
Procedure Note

Sierpiński Triangle Iteration

1. Open a new Sketchpad™ sketch.
2. Use the **Segment** tool to draw triangle *ABC*.
3. Select the vertices, and construct the triangle interior.
4. Select $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, in that order, and construct the midpoints *D, E,* and *F.*
5. Select the vertices again, and choose **Iterate** from the Transform menu. An Iterate dialog box will open. Select points *A, D,* and *F.* This maps triangle *ABC* onto the smaller triangle *ADF.*
6. In the Iterate dialog box, choose **Add A New Map** from the Structure pop-up menu. Map triangle *ABC* onto triangle *BED.*
7. Repeat Step 6 to map triangle *ABC* onto triangle *CFE.*
8. In the Iterate dialog box, choose **Final Iteration Only** from the Display pop-up menu.
9. In the Iterate dialog box, click Iterate to complete your construction.
10. Click in the center of triangle *ABC* and hide the interior to see your fractal at Stage 3.
11. Use Shift+plus or Shift+minus to increase or decrease the stage of your fractal.

Step 1 Follow the Procedure Note to create the Stage 3 Sierpiński triangle. The original triangle *ABC* is a Stage 0 Sierpiński triangle. Practice the last step of the Procedure Note to see how the fractal grows in successive stages. Write a sentence or two explaining what the Sierpiński triangle shows you about self-similarity.

Notice that the fractal's property of self-similarity does not change as you drag the vertices.



Step 2 What happens to the number of shaded triangles in successive stages of the Sierpiński triangle? Decrease your construction to Stage 1 and investigate. How many triangles would be shaded in a Stage *n* Sierpiński triangle? Use your construction and look for patterns to complete this table.

Stage number	0	1	2	3	...	*n*	...	50
Number of triangles	1	3			...		...	

What stage is the Sierpiński triangle shown on page 135?

Step 3 Suppose you start with a Stage 0 triangle (just a plain old triangle) with an area of 1 unit. What would be the shaded area at Stage 1? What happens to the shaded area in successive stages of the Sierpiński triangle? Use your construction and look for patterns to complete this table.

Stage number	0	1	2	3	...	n	...	50
Shaded area	1	$\frac{3}{4}$			...		...	

What would happen to the shaded area if you could infinitely increase the stage number?

Step 4 Suppose you start with a Stage 0 triangle with a perimeter of 6 units. At Stage 1 the perimeter would be 9 units (the sum of the perimeters of the three triangles, each half the size of the original triangle). What happens to the perimeter in successive stages of the Sierpiński triangle? Complete this table.

Stage number	0	1	2	3	...	n	...	50
Perimeter	6	9			...		...	

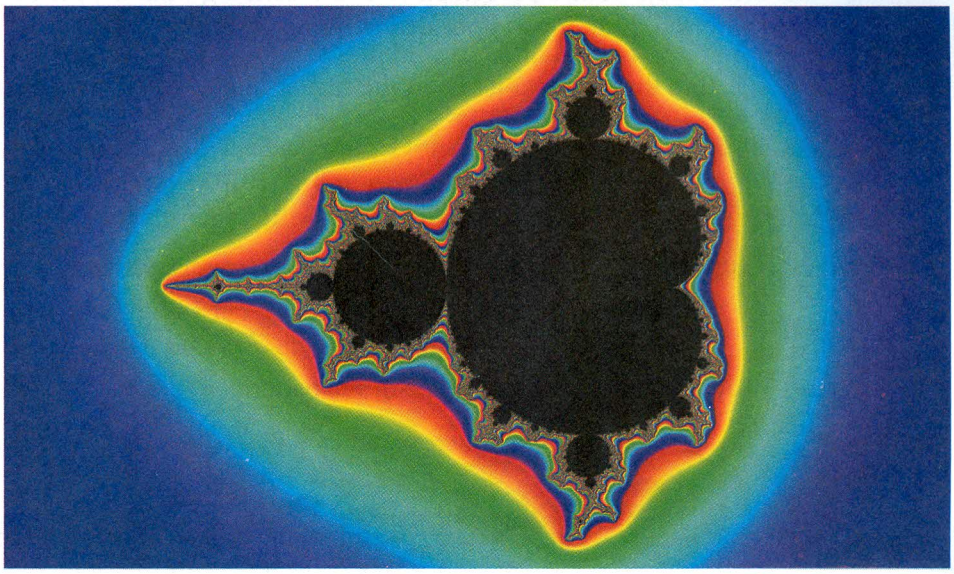
What would happen to the perimeter if you could infinitely increase the stage number?

Step 5 Increase your fractal to Stage 3 or 4. If you print three copies of your sketch, you can put the copies together to create a larger triangle one stage greater than your original. How many copies would you need to print in order to create a triangle two stages greater than your original? Print the copies you need and combine them into a poster or a bulletin board display.

Step 6 Sketchpad comes with a sample file of interesting fractals. Explore these fractals and see if you can use **Iterate** to create them yourself. You can save any of your fractal constructions by selecting the entire construction and then choosing **Create New Tool** from the Custom Tools menu. When you use your custom tool in the future, the fractal will be created without having to use **Iterate.**

The word *fractal* was coined by Benoit Mandelbrot (b 1924), a pioneering researcher in this new field of mathematics. He was the first to use high-speed computers to create the figure below, called the Mandelbrot set.

Only the black area is part of the set itself. The rainbow colors represent properties of points near the Mandelbrot set. To learn more about different kinds of fractals, visit www.keymath.com/DG.



CHAPTER 2 REVIEW

This chapter introduced you to inductive reasoning. You used inductive reasoning to observe patterns and make conjectures. You learned to disprove a conjecture with a counterexample and to explain why a conjecture is true with deductive reasoning. You learned how to predict number sequences with rules and how to use these rules to model application problems. Then you discovered special relationships about angle pairs and made your first geometry conjectures. Finally you explored the properties of corresponding, alternate interior, and alternate exterior angles formed by a transversal across parallel lines. As you review the chapter, be sure you understand all the important terms. Go back to the lesson to review any terms you're unsure of.

EXERCISES

1. "My dad is in the navy, and he says that food is great on submarines," said Diana. "My mom is a pilot," added Jill, "and she says that airline food is notoriously bad." "My mom is an astronaut trainee," said Julio, "and she says that astronauts' food is the worst imaginable." "You know," concluded Diana, "I bet no life exists beyond Earth! As you move farther and farther from the surface of Earth, food tastes worse and worse. At extreme altitudes, food must taste so bad that no creature could stand to eat. Therefore, no life exists out there." What do you think of Diana's reasoning? Is it inductive or deductive?

2. Think of a situation you observed outside of school in which inductive reasoning was used incorrectly. Write a paragraph or two describing what happened and explaining why you think it was poor inductive reasoning.

3. Think of a situation you observed outside of school in which deductive reasoning was used incorrectly. Write a paragraph or two describing what happened and explaining why you think it was poor deductive reasoning.

For Exercises 4–7, find the next two terms in the sequence.

4. 7, 21, 35, 49, 63, 77, ?, ?

5. Z, 1, Y, 2, X, 4, W, 8, ?, ? (h)

6. 7, 2, 5, −3, 8, −11, ?, ?

7. A, 4, D, 9, H, 16, M, 25, ?, ? (h)

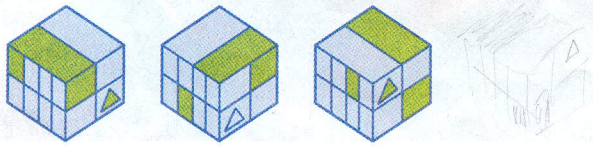
For Exercises 8 and 9, generate the first six terms in the sequence for each function rule.

8. $f(n) = n^2 + 1$

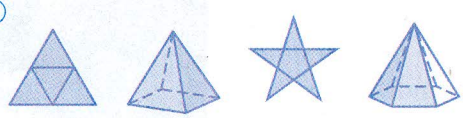
9. $f(n) = 2^{n-1}$ (h)

For Exercises 10 and 11, draw the next shape in the pattern.

10.



11. (h)



For Exercises 12–15, find the nth term and the 20th term in the sequence.

12.

n	1	2	3	4	5	6	...	n	...	20
$f(n)$	2	−1	−4	−7	−10	−13	...		...	

13.

n	1	2	3	4	5	6	...	n	...	20
$f(n)$	1	4	9	16	25	36	...		...	

14. ⓗ

n	1	2	3	4	5	6	...	n	...	20
$f(n)$	0	1	3	6	10	15	...		...	

15. ⓗ

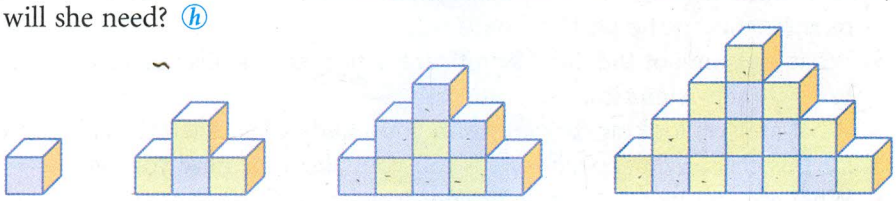
n	1	2	3	4	5	6	...	n	...	20
$f(n)$	1	3	6	10	15	21	...		...	

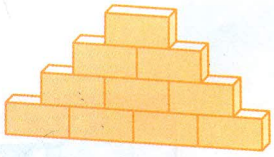
For Exercises 16 and 17, find a relationship. Then complete the conjecture.

16. Conjecture: The sum of the first 30 odd whole numbers is ?.

17. Conjecture: The sum of the first 30 even whole numbers is ?. ⓗ

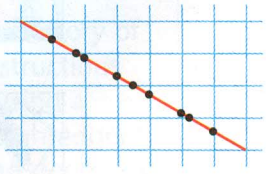
18. Viktoriya is a store window designer for Savant Toys. She plans to build a stack of blocks similar to the ones shown below but 30 blocks high. Make a conjecture for the value of the nth term and for the value of the 30th term. How many blocks will she need? ⓗ





19. The stack of bricks at right is four bricks high. Find the total number of bricks for a stack that is 100 bricks high.

20. For the 4-by-7 rectangular grid, the diagonal passes through 10 squares and 9 interior segments. In an 11-by-101 grid of squares, how many squares will the diagonal pass through? How many interior segments will it pass through? ⓗ



21. If at a party there are a total of 741 handshakes and each person shakes hands with everyone else at the party exactly once, how many people are at the party?

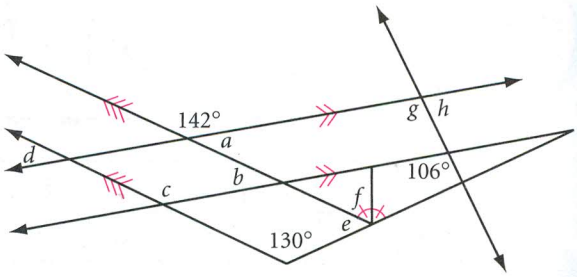
22. If 28 lines are drawn on a plane, what is the maximum number of points of intersection possible?

23. If a whole bunch of lines (no two parallel, no three concurrent) intersect in a plane 2926 times, how many lines are a whole bunch? ⓗ

24. If in a 54-sided polygon all possible diagonals are drawn from one vertex, they divide the interior of the polygon into how many regions?

25. How many sides does the polygon have if all possible diagonals drawn from one vertex divide the interior of the polygon into 54 regions?

26. Trace the diagram at right. Calculate each lettered angle measure.

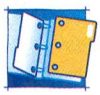


Assessing What You've Learned

WRITE IN YOUR JOURNAL

Many students find it useful to reflect on the mathematics they're learning by keeping a journal. Like a diary or a travel journal, a mathematics journal is a chance for you to reflect on what happens each day and your feelings about it. Unlike a diary, though, your mathematics journal isn't private—your teacher may ask to read it too, and may respond to you in writing. Reflecting on your learning experiences will help you assess your strengths and weaknesses, your preferences, and your learning style. Reading through your journal may help you see what obstacles you have overcome. Or it may help you realize when you need help.

- So far, you have written definitions, looked for patterns, and made conjectures. How does this way of doing mathematics compare to the way you have learned mathematics in the past?
- What are some of the most significant concepts or skills you've learned so far? Why are they significant to you?
- What are you looking forward to in your study of geometry? What are your goals for this class? What specific steps can you take to achieve your goals?
- What are you uncomfortable or concerned about? What are some things you or your teacher can do to help you overcome these obstacles?



KEEPING A NOTEBOOK You should now have four parts to your notebook: a section for homework and notes, an investigation section, a definition list, and now a conjecture list. Make sure these are up-to-date.

UPDATE YOUR PORTFOLIO Choose one or more pieces of your most significant work in this chapter to add to your portfolio. These could include an investigation, a project, or a complex homework exercise. Make sure your work is complete. Describe why you chose the piece and what you learned from it.